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Functional decompositions on vector-valued function spaces via operators

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ABSTRACT

Let X be a Banach space with a generalized basis. The Banach algebra $\mathcal{B}(X)$ of bounded linear operators on X is used to construct Banach spaces, $\mathcal{M}$ and $\mathcal{K}$, of weak* continuous functions from the state space of a C^*-algebra to $\mathcal{B}(X)$. If the basis satisfies certain properties, we prove that the dual space of $\mathcal{M}$ has a decomposition analogous to that of the dual space of $\mathcal{B}(X)$. In terms of the notion of M-ideal introduced by Alfsen and Effros, the subspace $\mathcal{K}$ is an M-ideal in the Banach space $\mathcal{M}$. For the cases of c_0 and ℓ^p, $1 < p < \infty$, we also prove an analogue of the result that $\operatorname{trace}(AB) = \operatorname{trace}(BA)$ for a trace class operator A and a bounded operator B on a Hilbert space.

1. Introduction

A beautiful theorem of Dixmier ([4], [10, p. 50, Theorem 5]) states that each bounded linear functional f on the algebra, $\mathcal{B}(\ell^2)$, of bounded linear operators on the Hilbert sequence space ℓ^2 admits a unique decomposition $f = g + h$, where g is a trace functional given by a trace class operator, and h is a *singular* functional, vanishing on the ideal, $\mathcal{K}(\ell^2)$, of compact operators. The most interesting part of Dixmier's Theorem is that $\|f\| = \|g\| + \|h\|$. This theorem has been generalized to much wider class of Banach spaces. Most significantly it has led to the vast literature of M-ideals, a notion introduced by Alfsen and Effros in 1972 [1]. For certain Banach space X, the bounded linear functionals on the algebra $\mathcal{B}(X)$ of bounded operators on X has the annihilators of the ideal, $\mathcal{K}(X)$, of compact operators on X as an ℓ^1 direct summand. That is $\mathcal{K}(X)$ is an M-ideal in $\mathcal{B}(X)$. Much of the recent work on M-ideals can be found in [5] and the references therein. Most of the known examples are the compact operators form an M-ideal in the space of bounded operators on certain Banach spaces (see [6,11,12,2,3]). Since operators on spaces with bases can be regarded as matrices with complex entries and C^*-algebras resemble the complex field in several different ways, it is natural to ask whether we can use operators to construct spaces of matrices with entries from a fixed C^*-algebra that contain M-ideals. Since the lack of commutativity can be a major obstacle, the use of functionals as tools seem to circumvent the difficulty. Here we use operators on Banach spaces with certain kind of bases to build examples of M-ideals in C^*-valued function spaces. That is with the complex field replaced by a C^*-algebra and using the state space rather than the norm as analytical tools we obtain generalizations of results on

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Banach space operators. Since the complex field $\mathbb{C}$ has only one state, namely the identity map on $\mathbb{C}$, our results become the decomposition of the dual of $\mathcal{B}(X)$. The methods used here are elementary.

We introduce notation and the class of Banach spaces with generalized basis, whose operators will be used to construct the C^*-valued function spaces, and then prove some preliminary results in Section 2. Section 3 contains results on operators on this special class of Banach spaces that will be used later. In Section 4 operators on Banach spaces with generalized bases will be used to define the spaces of functions on the Cartesian square of the indexing set of the basis of X taking values in a C^*-algebra (i.e., spaces of generalized C^*-matrices). In Sections 5 and 6 we prove the extension of Dixmier's Theorem on the C^*-valued function space. The one dimensional C^*-algebra case also subsumes all the cases of ℓ^p, $1 < p < \infty$ and c_0 (see [6]). With the C^*-algebra taken to be the complex field $\mathbb{C}$, the paper contains also an elementary proof of c_0 and ℓ^p, $1 < p < \infty$, analogues of $\operatorname{trace}(AB) = \operatorname{trace}(BA)$ for Hilbert space operators.

2. Notation, definitions and preliminaries

Let S be a nonempty set, and X a Banach space. Denote by $\mathcal{F}(S)$, or simply $\mathcal{F}$ when there is no ambiguity, the collection of all finite subsets of S directed by set inclusion. A function $\mathbf{x} : S \to X$ is said to be *summable* in X if there is an $x \in X$ to which the net (of "partial sums") $\{\sum_{s\in F} \mathbf{x}(s)\}_{F\in\mathcal{F}(s)}$ converges. If this is the case, x is called the *sum* of $\sum_{s\in S} \mathbf{x}(s)$, we say that $\sum_{s\in S} \mathbf{x}(s)$ converges in X, and write

$$\sum_{s\in S} \mathbf{x}(s) = \lim_{F\in\mathcal{F}(S)} \sum_{s\in F} \mathbf{x}(s) = x.$$

If no such an x exists, the sum $\sum_{s\in S} \mathbf{x}(s)$ *diverges* in X.

We omit the proof of the following useful Cauchy criterion.

Proposition 1. *Let X be a Banach space and $\mathbf{x} : S \to X$. Then the following conditions are equivalent.*

1. *$\sum_{s\in S} \mathbf{x}(s)$ converges in X.*
2. *For every $\epsilon > 0$, there is an $F_\epsilon \in \mathcal{F}$ such that*
$$\left\| \sum_{s\in G} \mathbf{x}(s) \right\| < \epsilon \quad \textit{for all } G \in \mathcal{F}(S \setminus F_\epsilon).$$
3. *For every $\epsilon > 0$ there is an $F \in \mathcal{F}$ such that*
$$\left\| \sum_{s\in S\setminus F} \mathbf{x}(s) \right\| < \epsilon.$$

A convergent sequence is always bounded, but, a convergent net, in general, may not be. However a convergent sum must have bounded "partial sums". This fact holds true for the more general class of topological vector spaces, see [13] for a proof of the following proposition.

Proposition 2. *Let X be a topological vector space. If $\sum_{s\in S} \mathbf{x}(s)$ converges in X, then the net $\{\sum_{s\in F} \mathbf{x}(s)\}_{F\in\mathcal{F}}$ (of "partial sums") is bounded in X.*

For each $G \subseteq S$ and each function $\mathbf{x}$ on S, $\mathbf{x}_G$ is the function $\mathbf{x}_G(s) = \mathbf{x}(s)$ if $s \in G$ and $\mathbf{x}(s) = 0$ otherwise.

For a fixed set $S \neq \emptyset$ and $p \in [1, \infty]$, $\ell^p(S)$ is the space of all functions $x : S \to \mathbb{C}$ such that $\|x\|_p^p = \sum_{s\in S} |x(s)|^p < \infty$, if $p < \infty$, and $\|x\|_\infty = \sup_{s\in S} |x(s)| < \infty$, if $p = \infty$. The generalized version of c_0 is the subspace of $\ell^\infty(S)$,

$$c_0(S) = \bigl\{x \in \ell^\infty(S)\colon (\forall\epsilon > 0) \bigl(\exists\, F \in \mathcal{F}(S)\bigr) (\forall s \in S \setminus F) \bigl(\bigl|x(s)\bigr| < \epsilon\bigr)\bigr\}.$$

These are generalized sequence spaces, and are Banach spaces with the norm as defined.

The notion of Schauder bases is inspired by the standard basis,

$$\bigl\{e_n = \{\underbrace{0, \ldots, 0, 1}_{n}, 0 \ldots\}\colon n \in \mathbb{N}\bigr\}$$

in ℓ^p sequence spaces for $1 \leqslant p < \infty$. Analogously, the spaces $c_0(S)$ and $\ell^p(S)$ with $p \in [1, \infty)$ have the standard "generalized basis" $\{e_s\colon s \in S\}$, where $e_s : S \to \mathbb{C}$ is defined by $e_s(t) = \delta_{st}$ $\forall t \in S$ ($\delta_{st} = 0$ if $s \neq t$ and $\delta_{ss} = 1$).

A set of vectors $\{e_s\colon s \in S\}$ in a Banach space X is called a *generalized basis*, or *g-basis* of X if for each $x \in X$, there is a *unique* function $f_x : S \to \mathbb{C}$ such that

$$x = \sum_{s\in S} f_x(s) e_s.$$

Since the convergence here is unconditional convergence, so when the set S is taken to be the set of natural numbers $\mathbb{N}$, it is exactly the unconditional Schauder basis [8, Section 4.2].

As nonzero multiples of vectors in a g-basis form another g-basis, we will assume that each vector in a g-basis is a unit vector. We will also identify each vector x in X with its representing function f_x, and we will simply treat x as the function f_x, though in some situations we may need to get back to the actual sum $x = \sum_{s\in S} x(s)e_s$. In essence, a g-basis converts X into a space of complex valued functions defined on S, the indexing set of the g-basis.

We omit the proofs of the following propositions.

Proposition 3. *Let $\{e_s\}_{s\in S}$ be a g-basis for X. Then*

$$|||x||| := |||x|||_{\{e_s\}_{s\in S}} := \sup_{F\in\mathcal{F}} \left\| \sum_{s\in F} f_x(s)e_s \right\|, \quad x \in X$$

is an equivalent norm on X and $|||x||| \geqslant \|x\|$ for all $x \in X$.

Proposition 4. *Let $\{e_s\}_{s\in S}$ be a g-basis for X.*

1. *For each subset S_0 of S, let P_{S_0} be defined by*

$$P_{S_0}(x) = \sum_{s\in S_0} f_s(x)e_s, \quad \text{for all } x \in X.$$

Then P_{S_0} is a bounded projection.

2. *For each $s_0 \in S$, then the functional f_{s_0} defined by*

$$f_{s_0}(x) := f_x(s_0) \quad (x \in X)$$

is a bounded linear functional on X.

Motivated by the norm on the classical sequence spaces, we also further assume that the norm on X (treating X as a space of complex-valued functions on S) has the *N-property* as defined below (see [6]).

Definition 5. The norm on a Banach space $\mathcal{X}$ of complex valued functions on the set S is said to have the *N-property* if there is a nonnegative real-valued function N on $[0,\infty)\times[0,\infty)$ such that

1. $N(\alpha,\beta) \leqslant N(a,b)$ for all $0 \leqslant \alpha \leqslant b$ and $0 \leqslant \beta \leqslant b$; and
2. for all $x \in \mathcal{X}$,

$$\|x\| = N\big(\|x_F\|, \|x_{S\setminus F}\|\big) \quad \forall F \in \mathcal{F}.$$

For condition (2) above, we have implicitly used the facts that $x_F \in \mathcal{X}$ since F is finite and $x_{S\setminus F} = x - x_F \in \mathcal{X}$, for all $x \in \mathcal{X}$ and $F \in \mathcal{F}$. Clearly, the generalized sequence spaces $c_0(S)$ and $\ell^p(S)$ with $1 \leqslant p < \infty$ with their standard g-basis have the N-property with $N(a,b) = \max\{a,b\}$ and $N(a,b) = (a^p + b^p)^{1/p}$ respectively. A Banach space with a g-basis can also have the N-property when the space is treated as a space of complex valued functions on the indexing set of the g-basis.

Parts 5, 2 and 1 of the following lemma are the stronger versions of Propositions 3 and 4 under the N-property assumption.

Lemma 6. *Let X be a Banach space with a basis $\{e_s\}_{s\in S}$ having the N-property and $x \in X$.*

1. *For each $s \in S$, $|x(s)| \leqslant \|x\|$.*
2. *For each $F \in \mathcal{F}(S)$, $\|x_F\| \leqslant \|x\|$.*
3. *The net $\{\|x_F\|\}_{F\in\mathcal{F}}$ is increasing.*
4. *If $G \subseteq H \subseteq S$, then $x_G, x_H \in X$, and $\|x_G\| \leqslant \|x_H\| \leqslant \|x\|$.*
5. $\|x\| = \sup_{F\in\mathcal{F}(S)} \|x_F\| = \lim_{F\in\mathcal{F}(S)} \|x_F\|$.
6. *If $f, g \in \mathbb{C}^S$ satisfy $|f(s)| \leqslant |g(s)|$ for all $s \in S$, and if*

$$\sum_{s\in S} g(s)e_s \quad \text{converges in } X,$$

then

$$\sum_{s\in S} f(s)e_s \quad \textit{converges in } X, \quad \textit{and}$$

$$\left\|\sum_{s\in G} f(s)e_s\right\| \leqslant \left\|\sum_{s\in G} g(s)e_s\right\| \quad \textit{for all } G \subseteq S.$$

7. *For each $s \in S$, there is an $e_s^{\#} \in X^{\#}$ such that $e_s^{\#}(e_t) = \delta_{st}$ for all $t \in S$.*

Proof. We prove (2), then (1) follows. From the N-property assumption,

$$\|x\| = \|x_F + x_{S\setminus F}\| = N\big(\|x_F\|, \|x_{S\setminus F}\|\big) \geqslant N\big(\|x_F\|, 0\big) = N\big(\big\|(x_F)_F\big\|, \big\|(x_F)_{S\setminus F}\big\|\big) = \|x_F\|.$$

(3) Let $F, G \in \mathcal{F}(S)$ be such that $F \subseteq G$. Then

$$\|x_G\| = \big\|(x_G)_F + (x_G)_{G\setminus F}\big\| = N\big(\|x_F\|, \|x_{G\setminus F}\|\big) \geqslant N\big(\|x_F\|, 0\big) = \|x_F\|.$$

(4) Let $\epsilon > 0$. Since $x \in X$, by Cauchy criterion (Proposition 1), there is an $F_\epsilon \in \mathcal{F}$ such that

$$\|x_F\| < \epsilon \quad \text{for all } F \in \mathcal{F}(S \setminus F_\epsilon).$$

Let $F_1 = F_\epsilon \cap G$. Then $F_1 \in \mathcal{F}(G)$. Let $F \in \mathcal{F}(G \setminus F_1)$. Then $F \subseteq S \setminus F_\epsilon$, and hence $\|x_F\| < \epsilon$. Since ϵ is arbitrary, by Cauchy criterion,

$$x_G = \sum_{s\in G} x(s)e_s \quad \text{converges in } X.$$

Let $G \subseteq H \subseteq S$; and let $\eta > 0$. There is an $F_\eta \in \mathcal{F}(G)$ such that

$$\|x_G - x_F\| < \eta \quad \text{for all } F_\eta \subseteq F \in \mathcal{F}(G).$$

Let $F_\eta \subseteq F \in \mathcal{F}(G)$. Then, $F, F_\eta \in \mathcal{F}(H)$, and hence, by part (2), with $x_H(s) = x(s)$ if $s \in H$, and $x_H(s) = 0$ if $s \in S \setminus H$,

$$\|x_H\| = \left\|\sum_{s\in S} x_H(s)e_s\right\| \geqslant \left\|\sum_{s\in F} x_H(s)e_s\right\| = \left\|\sum_{s\in F} x(s)e_s\right\| = \|x_F\| \geqslant \|x_G\| - \|x_G - x_F\| > \|x_G\| - \eta.$$

Since η is arbitrary, $\|x_H\| \geqslant \|x_G\|$ for $S \supseteq H \supseteq G$. This also shows that $\|x\| \geqslant \|x_H\|$ for $S \supseteq H$.

(5) Let $\epsilon > 0$. Since $x = \sum_{s\in S} x(s)e_s$ converges in X, there is an $F_\epsilon \in \mathcal{F}(S)$ such that

$$\left\|x - \sum_{s\in F} x(s)e_s\right\| < \epsilon \quad \text{for all } F \in \mathcal{F}(S),\ F \supseteq F_\epsilon.$$

Thus

$$\sup_{F\in\mathcal{F}(S)} \|x_F\| \geqslant \|x_{F_\epsilon}\| = \big\|x - (x - x_{F_\epsilon})\big\| \geqslant \|x\| - \|x - x_{F_\epsilon}\| > \|x\| - \epsilon.$$

(6) Let $G \subseteq S$. Part (4) implies that $\sum_{s\in G} g(s)e_s$ converges. We show inductively that the inequality holds for all finite subsets of G. For a singleton set $F = \{s_1\} \in \mathcal{F}(G)$,

$$\big\|f(s_1)e_1\big\| = \big|f(s_1)\big|\|e_1\| \leqslant \big|g(s_1)\big|\|e_1\| = \big\|g(s_1)e_1\big\|.$$

Suppose for each k element set $F = \{t_1, t_2, \ldots, t_k\} \in \mathcal{F}(G)$, we have

$$\big\|f(t_1)e_{t_1} + \cdots + f(t_k)e_{t_k}\big\| \leqslant \big\|g(t_1)e_{t_1} + \cdots + f(t_k)e_{t_k}\big\|.$$

Let $\{s_1, s_2, \ldots, s_{k+1}\} \subseteq G$. Then

$$\begin{aligned}\big\|f(s_1)e_{s_1} + \cdots + f(s_{k+1})e_{s_{k+1}}\big\| &= N\big(\big\|f(s_1)e_{s_1}\big\|, \big\|f(s_2)e_2 + \cdots + f(s_{k+1})e_{k+1}\big\|\big)\\ &\leqslant N\big(\big\|g(s_1)e_{s_1}\big\|, \big\|g(s_2)e_2 + \cdots + g(s_{k+1})e_{k+1}\big\|\big)\\ &= \big\|g(s_1)e_{s_1} + g(s_2)e_2 + \cdots + g(s_{k+1})e_{k+1}\big\|.\end{aligned}$$

This shows that the inequality holds for all $k+1$-element subsets of G whenever it holds for all k-element subsets of G. Since it holds for all singleton subsets of G, we conclude, by induction, that

$$\left\|\sum_{s\in F} f(s)e_s\right\| \leqslant \left\|\sum_{s\in F} g(s)e_s\right\| \quad \forall F \in \mathcal{F}(G).$$

Thus, by part (5), for all $G \subseteq S$ and all $F \in \mathcal{F}(G)$,

$$\Big\|\sum_{s\in F} f(s)e_s\Big\| \leqslant \Big\|\sum_{s\in F} g(s)e_s\Big\| \leqslant \sup_{H\in\mathcal{F}(G)}\Big\|\sum_{s\in H} g(s)e_s\Big\| = \Big\|\sum_{s\in G} g(s)e_s\Big\|,$$

and hence

$$\Big\|\sum_{s\in G} f(s)e_s\Big\| = \sup_{F\in\mathcal{F}(G)}\Big\|\sum_{s\in F} f(s)e_s\Big\| \leqslant \Big\|\sum_{s\in G} g(s)e_s\Big\| \quad \forall G \subseteq S.$$

(7) Let X_0 be the subspace of X consisting of all *finite* linear combinations of elements (i.e., algebraic linear span) of the set $\{e_s\colon\ s \in S\}$. For each $s \in S$, define $g_s(e_t) = \delta_{st}$, and extend linearly to all of X_0. Each $x \in X_0$ corresponds to a unique $F_x \in \mathcal{F}(S)$ such that $x = \sum_{t\in F_x} x(t)e_t$. Define $x(t) = 0$ for all $t \in S \setminus F_x$. Then $x = \sum_{t\in S} x(t)e_t$ is the g-basis representation of x in X. Thus $|g_s(x)| = |x(s)| \leqslant \|x\|$, by part (1), for all $x \in X_0$. By Hahn–Banach Theorem, there is an extension $e_s^\#$ on X such that $\|e_s^\#\| = \|g_s\| = 1$. $\square$

The collection $\{e_s, e_s^\#\}_{s\in S}$ of basis vectors and their dual functionals is a *biorthogonal system*. If the system $\{e_s^\#\}_{s\in S}$ is also a g-basis for $X^\#$, it is the basis dual to $\{e_s\}_{s\in S}$. Some examples of dual pairs of spaces with dual bases are

$$\ell^p(S,\mathbb{C}) \quad\text{and}\quad \big[\ell^p(S,\mathbb{C})\big]^\# \cong \ell^q(S,\mathbb{C}), \quad 1 < p < \infty,\ \frac{1}{p} + \frac{1}{q} = 1.$$

So is the pair

$$c_0(S,\mathbb{C}) \quad\text{and}\quad \big[c_0(S,\mathbb{C})\big]^\# \cong \ell^1(S,\mathbb{C}).$$

3. Operators on spaces with generalized bases

In this section, we will represent operators on a Banach space with a g-basis that makes the norm satisfies the N-property (Definition 5) as generalized matrices, and construct some special compact operators to be used later.

Theorem 7. *Let X be a Banach space with a g-basis $\{e_s\}_{s\in S}$ such that the norm satisfies the N-property. (Each vector in X is identified with its representing complex-valued function on S.)*

1. *If $A \in \mathbb{C}^{S\times S}$ satisfies*
   (a) *for each $x \in X$, $(Ax)(s) := \sum_{t\in S}(A(s,t))x(t)$ converges for every $s \in S$; and*
   (b) *the function Ax is in X (i.e. $\sum_{s\in S}(Ax)(s)e_s$ converges in X);*
   then $A : x \mapsto Ax$ defines a bounded linear operator on X.
2. *For every bounded linear operator T on X, there is a unique function $A_T : S \times S \to \mathbb{C}$ such that*

$$(Tx)(s) = \sum_{t\in S}\big[A_T(s,t)\big]x(t) \quad \textit{for all } x \in X, \textit{ and all } s \in S.$$

Proof. A C^*-valued version of part (1) can be found in [13], so we omit the proof, and prove here only part (2). Let $t \in S$. Since $T(e_t) \in X$, there is a complex-valued function f_t on S such that

$$T(e_t) = \sum_{s\in S}\big(f_t(s)\big)e_s.$$

For each $(s,t) \in S \times S$, define $A_T(s,t) = f_t(s)$. Let $\{e_s^\#\}$ be the collection of bounded linear functionals on X associated with $\{e_s\}$ as in as in Lemma 6(7). For each $x \in X$, since

$$x = \sum_{t\in S} x(t)e_t \quad \text{converges in } X,$$

(using the conventional notation $\langle y, h\rangle = h(y)$ for $y \in X$, $h \in X^\#$) we have

$$\langle x, e_u^\#\rangle = \Big\langle \lim_{F\in\mathcal{F}} \sum_{t\in F} x(t)e_t, e_u^\#\Big\rangle = \lim_{F\in\mathcal{F}}\Big\langle \sum_{t\in F} x(t)e_t, e_u^\#\Big\rangle = x(u)$$

for all $u \in S$. Thus

$$x = \sum_{t\in S} x(t)e_t = \sum_{t\in S}\langle x, e_t^\#\rangle e_t.$$

By continuity of T,

$$Tx = T\left(\lim_{F\in\mathcal{F}(S)}\left[\sum_{t\in F(S)} x(t)e_t\right]\right) = \lim_{F\in\mathcal{F}(S)}\left[T\left(\sum_{t\in F} x(t)e_t\right)\right] = \lim_{F\in\mathcal{F}(S)}\left[\sum_{t\in F} x(t)Te_t\right]$$
$$= \lim_{F\in\mathcal{F}(S)}\left[\sum_{t\in F} x(t)\left(\sum_{s\in S} f_t(s)e_s\right)\right] = \lim_{F\in\mathcal{F}(S)}\left[\sum_{s\in S}\left(\sum_{t\in F} A_T(s,t)x(t)\right)e_s\right],$$

and hence, for each $u \in S$,

$$\langle Tx, e_u^{\#}\rangle = \left\langle \lim_{F\in\mathcal{F}(S)}\left[\sum_{s\in S}\left(\sum_{t\in F} A_T(s,t)x(t)\right)e_s\right], e_u^{\#}\right\rangle = \lim_{F\in\mathcal{F}(S)}\left\langle\left[\sum_{s\in S}\left(\sum_{t\in F} A_T(s,t)x(t)e_s\right)\right], e_u^{\#}\right\rangle$$
$$= \lim_{F\in\mathcal{F}(S)}\left\langle\left[\lim_{G\in\mathcal{F}(S)}\sum_{s\in G}\left(\sum_{t\in F} A_T(s,t)x(t)e_s\right)\right], e_u^{\#}\right\rangle = \lim_{F\in\mathcal{F}(S)}\left[\lim_{G\in\mathcal{F}(S)}\left\langle\sum_{s\in G}\left(\sum_{t\in F} A_T(s,t)x(t)e_s\right), e_u^{\#}\right\rangle\right]$$
$$= \lim_{F\in\mathcal{F}(S)}\left(\sum_{t\in F} A_T(u,t)x(t)\right) = \sum_{t\in S} A_T(u,t)x(t).$$

Thus

$$Tx = \sum_{s\in S}\langle Tx, e_s^{\#}\rangle e_s = \sum_{s\in S}\left(\sum_{t\in S} A_T(s,t)x(t)\right)e_s \quad \text{for all } x \in X;$$

that is

$$(Tx)(s) = \sum_{t\in S} A_T(s,t)x(t) \quad \text{for all } s \in S, \text{ for each } x \in X. \qquad \square$$

We will call the representing function A_T of the operator T the *g-matrix representation* of T with respect to the g-basis $\{e_s\}_{s\in S}$. Part (1) of Theorem 7 says that whenever a function on $S \times S$ defines an operator on X, that is it multiplies each vector of X to a vector in X, then it defines a bounded linear operator on X.

We will assume that X and $X^{\#}$ have biorthogonal g-bases $\{e_s, e_s^{\#}\}_{s\in S}$, where $\{e_s^{\#}\}$ is the set of functionals constructed from the basis $\{e_s\}$ of X as in Lemma 6(7). Each bounded linear operator T on X will be identified with its representing function, A_T (as in part (2) of Theorem 7). For each $G \subseteq S$, $A_{\underline{G}}$, $A_{G|}$, and $A_{G_\lrcorner}$ are the functions defined, respectively, by

$$A_{\underline{G}}(s,t) = \begin{cases} A(s,t) & \text{if } s \in G, \\ 0 & \text{otherwise,} \end{cases} \qquad A_{G|}(s,t) = \begin{cases} A(s,t) & \text{if } t \in G, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$A_{G_\lrcorner} = (A_{\underline{G}})_{G|}(s,t) = (A_{G|})_{\underline{G}}(s,t) = \begin{cases} A(s,t) & \text{if } s,t \in G, \\ 0 & \text{otherwise.} \end{cases}$$

We include the following theorems for convenience of reference and their proofs for completeness.

Lemma 8. *Let $A \in \mathcal{B}(X)$. Then, for each $G \subseteq S$,*

1. $A_{\underline{G}} \in \mathcal{B}(X)$, $\|A_{\underline{G}}\| \leqslant \|A\|$, *and* $\{\|A_{\underline{F}}\|\}_{F\in\mathcal{F}(S)}$ *increases to* $\|A\|$.
2. $A_{G|} \in \mathcal{B}(X)$, $\|A_{G|}\| \leqslant \|A\|$, *and* $\{\|A_{F|}\|\}_{F\in\mathcal{F}(S)}$ *increases to* $\|A\|$.
3. $A_{G_\lrcorner} \in \mathcal{B}(X)$, $\|A_{G_\lrcorner}\| \leqslant \|A\|$, *and* $\{\|A_{F_\lrcorner}\|\}_{F\in\mathcal{F}(S)}$ *increases to* $\|A\|$.

Proof. (1) Let $x \in X$. By Lemma 6(4),

$$\|A_{\underline{G}}x\| = \left\|(Ax)_G\right\| \leqslant \|Ax\| \leqslant \|A\|\|x\|$$

and hence $A_{\underline{G}} \in \mathcal{B}(X)$ with $\|A_{\underline{G}}\| \leqslant \|A\|$. Let $\eta > 0$. There is a unit vector $x \in X$ such that

$$\left\|(A_{\underline{G}})x\right\| = \left\|(Ax)_G\right\| > \|A_{\underline{G}}\| - \eta.$$

Thus, for $G \subseteq H \subseteq S$, by Lemma 6 again,

$$\|A_{\underline{H}}\| \geqslant \left\|(A_{\underline{H}})x\right\| = \left\|(Ax)_H\right\| \geqslant \left\|(Ax)_G\right\| > \|A_{\underline{G}}\| - \eta.$$

Since η is arbitrary, $\|A_{\underline{H}}\| \geqslant \|A_{\underline{G}}\|$.

Let $\epsilon > 0$. There is a unit vector $x \in X$ such that

$$\|A\| - \frac{\epsilon}{2} < \|Ax\| = \left\| \sum_{s \in S} \left[\sum_{t \in S} A(s,t)x(t) \right] e_s \right\|.$$

From the convergence, there exists $F_0 \in \mathcal{F}(S)$ such that

$$\left\| \sum_{s \in F} \left[\sum_{t \in S} A(s,t)x(t) \right] e_s \right\| > \|Ax\| - \frac{\epsilon}{4} \quad \text{for all } F_0 \subseteq F \in \mathcal{F}(S).$$

Thus, for each $F_0 \subseteq F \in \mathcal{F}(S)$,

$$\|A_{\underline{F}}\| \geqslant \left\|(A_{\underline{F}})x\right\| = \left\| \sum_{s \in F} \left[\sum_{t \in S} A(s,t)x(t) \right] e_s \right\| > \|Ax\| - \frac{\epsilon}{4} > \|A\| - \epsilon.$$

This shows that

$$\lim_{F \in \mathcal{F}(S)} \|A_F\| = \|A\|.$$

(2) Since X and $X^{\#}$ are assumed to have biorthogonal g-bases, a routine calculation reveals that

$$A(s,t) = A^{\#}(t,s) \quad \text{for all } (s,t) \in S \times S, \quad \text{and} \quad \left\|A^{\#}\right\| = \|A\|.$$

Thus, for $G \subseteq H \subseteq S$,

$$\|A_{G|}\| = \left\|\left[(A^{\#})_{\underline{G}}\right]^{\#}\right\| = \left\|(A^{\#})_{\underline{G}}\right\| \leqslant \left\|(A^{\#})_{\underline{H}}\right\| = \left\|\left[(A^{\#})_{\underline{H}}\right]^{\#}\right\| = \|A_{H|}\|.$$

(3) For $G \subseteq H \subseteq S$, by parts (1) and (2),

$$\|A_{G_\lrcorner}\| = \left\|(A_{\underline{G}})_{G|}\right\| \leqslant \left\|(A_{\underline{G}})_{H|}\right\| = \left\|(A_{H|})_{\underline{G}}\right\| \leqslant \left\|(A_{H|})_{\underline{H}}\right\| = \|A_{H_\lrcorner}\|.$$

With ϵ, x, and F_0 as in the last paragraph of part (1), from the finiteness of F_0, there is a finite $F_1 \supseteq F_0$ such that

$$\left| \sum_{t \in F} A(s,t)x(t) \right| > \left| \sum_{t \in S} A(s,t)x(t) \right| - \frac{\epsilon}{4\,\mathrm{Card}(F_0) + 1}$$

for all $F_1 \subseteq F \in \mathcal{F}(S)$, and all $s \in F_0$. Let $F_1 \subseteq F \in \mathcal{F}(S)$. Then $F \supseteq F_0$, and, by Lemma 6 (6),

$$\left\| \sum_{s \in F} \left[\sum_{t \in F} A(s,t)x_t \right] e_s \right\| \geqslant \left\| \sum_{s \in F_0} \left[\left| \sum_{t \in S} A(s,t)x(t) \right| - \frac{\epsilon}{4\,\mathrm{Card}(F_0) + 1} \right] e_s \right\| \geqslant \left\| \sum_{s \in F_0} \left| \sum_{t \in S} A(s,t)x(t) \right| e_s \right\| - \frac{\epsilon}{4}.$$

Hence

$$\begin{aligned} \|A_{F_\lrcorner}\| + \frac{\epsilon}{4} &\geqslant \left\|(A_{F_\lrcorner})x\right\| + \frac{\epsilon}{4} = \left\| \sum_{s \in F} \left[\sum_{t \in F} A(s,t)x(t) \right] e_s \right\| + \frac{\epsilon}{4} \\ &\geqslant \left\| \sum_{s \in F_0} \left| \sum_{t \in S} A(s,t)x(t) \right| e_s \right\| \quad \bigl(\text{using Lemma 6(6) twice}\bigr) \\ &= \left\| \sum_{s \in F_0} \left[\sum_{t \in S} A(s,t)x(t) \right] e_s \right\| > \|Ax\| - \frac{\epsilon}{4} > \|A\| - \frac{3\epsilon}{4}, \end{aligned}$$

i.e., $\|A_{F_\lrcorner}\| > \|A\| - \epsilon$ for all $F_1 \subseteq F \in \mathcal{F}(S)$. Therefore $\lim_{F \in \mathcal{F}(S)} \|A_{F_\lrcorner}\| = \|A\|$. $\square$

An operator K on X is said to be *compact* if $K[X]_1$ is totally bounded; i.e., the image of the closed unit ball under K has compact closure. The following is a collection of folklore results, and, perhaps, well known or even trivial to the experts. Again, we include the proofs for completeness and convenience of reference.

Lemma 9. *Let X and $X^{\#}$ have biorthogonal g-bases $\{e_s, e_s^{\#}\}_{s \in S}$; and let $A \in \mathcal{B}(X)$. Then the following are equivalent.*

1. *A is compact.*
2. *There exists a sequence $\{F_n\} \subseteq \mathcal{B}(X)$ of bounded finite rank operators such that*

$$\lim_{n \to \infty} \|A - F_n\| = 0.$$

3. $\lim_{F \in \mathcal{F}} \|A - A_{\underline{F}}\| = 0$.

4. $\lim_{G\in\mathcal{F}(S)} \|A - A_{G|}\| = 0$.
5. $\lim_{F\in\mathcal{F}(S)} \|A - A_{F_\lrcorner}\| = 0$.
6. $\{A_{G|}\}_{G\in\mathcal{F}(S)}$ *is a Cauchy net in* $\mathcal{B}(X)$.
7. $\{A_{\underline{F}}\}_{F\in\mathcal{F}(S)}$ *is a Cauchy net in* $\mathcal{B}(X)$.

Proof. [(1) $\Rightarrow$ (3)] Let $\epsilon > 0$. Since $\{B(Ax, \frac{\epsilon}{4})\colon\ x \in X,\ \|x\| \leqslant 1\}$ is an open cover for the closure of the set $C := \{Ax\colon\ x \in [X]_1\} = A[X]_1$, there exist $x_1, \dots x_k \in [X]_1$, such that

$$C \subseteq \bigcup_{j=1}^{k} B\left(Ax_j, \frac{\epsilon}{4}\right).$$

Since

$$\lim_{F\in\mathcal{F}(S)} \left\|(A - A_{\underline{F}})x_j\right\| = \lim_{F\in\mathcal{F}(S)} \left\|Ax_j - (Ax_j)_F\right\| = 0 \quad \text{for each } j = 1, \dots, k,$$

by definition of g-basis, there is an $F_0 \in \mathcal{F}(S)$ such that

$$\left\|(A - A_{\underline{F}})x_j\right\| < \frac{\epsilon}{4} \quad \text{for all } F_0 \subseteq F \in \mathcal{F}(S), \text{ and all } 1 \leqslant j \leqslant k.$$

Let $F_0 \subseteq F \in \mathcal{F}(S)$; and let $x \in [X]_1$. There is an l with $1 \leqslant l \leqslant k$ such that

$$\|Ax - Ax_l\| < \frac{\epsilon}{4}.$$

Thus

$$\left\|(A - A_{\underline{F}})x\right\| \leqslant \|Ax - Ax_l\| + \|Ax_l - A_{\underline{F}}x_l\| + \|A_{\underline{F}}x_l - A_{\underline{F}}x\| < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \left\|[Ax_l - Ax]_{\underline{F}}\right\| \leqslant \frac{\epsilon}{2} + \|Ax_l - Ax\| < \frac{3\epsilon}{4}.$$

Since x is arbitrary from the closed unit ball $[X]_1$ of X, this implies that

$$\|A - A_{\underline{F}}\| \leqslant \frac{3\epsilon}{4} < \epsilon.$$

Since F can be any finite subset of S containing F_0, we have proven that

$$\lim_{F\in\mathcal{F}(S)} \|A - A_{\underline{F}}\| = 0.$$

[(3) $\Rightarrow$ (2)] This is clear since each $A_{\underline{F}}$, with $F \in \mathcal{F}(S)$, is a bounded finite rank operator: $(A_{\underline{F}}x)(s) = 0$ for all $x \in X$ and all $s \in S \setminus F$.

[(2) $\Rightarrow$ (1)] Let $\epsilon > 0$. There is a finite rank operator $K \in \mathcal{B}(X)$ such that $\|A - K\| < \frac{\epsilon}{4}$. Since the set $V = \{Kx\colon\ x \in [X]_1\}$ is a bounded subset of a finite dimensional space, it is totally bounded. Thus there are vectors $v_1, v_2, \dots, v_n \in [X]_1$ such that

$$V \subseteq \bigcup_{j=1}^{n} B\left(Kv_j, \frac{\epsilon}{4}\right).$$

Let $x \in [X]_1$. There is a $j = 1, 2, \dots, n$ such that $\|Kx - Kv_j\| < \frac{\epsilon}{4}$. Thus

$$\|Ax - Av_j\| \leqslant \|Ax - Kx\| + \|Kx - Kv_j\| + \|Kv_j - Av_j\| \leqslant \|A - K\| + \frac{\epsilon}{4} + \|A - K\| < \frac{3\epsilon}{4} < \epsilon.$$

This shows that

$$A\big[[X]_1\big] \subseteq \bigcup_{j=1}^{n} B(Av_j, \epsilon).$$

[(1) $\Rightarrow$ (4)] Assume A is compact. Then the adjoint $A^{\#}$ of A is compact. Since $\|B^{\#}\| = \|B\|$ for each $B \in \mathcal{B}(X)$, we conclude from the proven equivalence of (1) and (3) that,

$$\lim_{F\in\mathcal{F}(S)} \|A - A_{F|}\| = \lim_{F\in\mathcal{F}(S)} \left\|(A - A_{F|})^{\#}\right\| = \lim_{F\in\mathcal{F}(S)} \left\|A^{\#} - \left(A^{\#}\right)_{\underline{F}}\right\| = 0.$$

[(4) $\Rightarrow$ (5)] Let $\epsilon > 0$. There is an $F_0 \in \mathcal{F}(S)$ such that

$$\|A - A_{F|}\| < \frac{\epsilon}{6} \quad \text{for all } F_0 \subseteq F \in \mathcal{F}(S). \tag{1}$$

Set $K = A_{F_0|}$. By definitions of g-basis and $K_{\underline{F}}$,

$$\lim_{F\in\mathcal{F}(S)} \|Ke_s - K_{\underline{F}}e_s\| = 0 \quad \text{for all } s \in S.$$

From the finiteness of F_0, there is a $G_0 \in \mathcal{F}(S)$ such that

$$\|Ke_s - K_{\underline{G}}e_s\| < \frac{\epsilon}{6\,\mathrm{Card}(F_0)+1} \quad \text{for all } s \in F_0 \text{ and } G_0 \subseteq G \in \mathcal{F}(S).$$

Let

$$x = \sum_{s\in S} x(s)e_s \in [X]_1.$$

Then by Lemma 6 part (1), for each $t \in S$,

$$\big|x(t)\big| = \big\|x(t)e_t\big\| \leqslant \left\|\sum_{s\in S} x(s)e_s\right\| = \|x\| \leqslant 1.$$

For $G_0 \subseteq G \in \mathcal{F}(S)$, since $(K - K_{\underline{G}})e_s = 0$ for all $s \in S \setminus F_0$, from Lemma 6 part (6) we have,

$$\big\|(K - K_{\underline{G}})x\big\| \leqslant \sum_{s\in F_0} \big|x(s)\big|\big\|(K - K_{\underline{G}})e_s\big\| \leqslant \sum_{s\in F_0} \big\|(K - K_{\underline{G}})e_s\big\| < \frac{\epsilon}{6}.$$

Since $x \in [X]_1$ is arbitrary, we conclude that

$$\|K - K_{\underline{G}}\| \leqslant \frac{\epsilon}{6} \quad \text{for all } G_0 \subseteq G \in \mathcal{F}(S). \tag{2}$$

Let $F_1 = F_0 \cup G_0$. Then, if $F_1 \subseteq F \in \mathcal{F}(S)$, first we have, from inequality (1)

$$\|A_{F|} - K\| = \|A_{F|} - A_{F_0|}\| \leqslant \|A_{F|} - A\| + \|A - A_{F_0|}\| < \frac{\epsilon}{3}. \tag{3}$$

Thus, by inequalities (1), (2), and (3),

$$\begin{aligned}
\|A - A_{F\lrcorner}\| &\leqslant \|A - A_{F|}\| + \|A_{F|} - A_{F\lrcorner}\| < \frac{\epsilon}{6} + \big\|A_{F|} - (A_{F|})_{\underline{F}}\big\| \\
&\leqslant \frac{\epsilon}{6} + \|A_{F|} - A_{F_0|}\| + \big\|A_{F_0|} - (A_{F_0|})_{\underline{F}}\big\| + \big\|(A_{F_0|})_{\underline{F}} - (A_{F|})_{\underline{F}}\big\| \\
&< \frac{\epsilon}{6} + \frac{\epsilon}{3} + \|K - K_{\underline{F}}\| + \big\|(A_{F_0|} - A_{F|})_{\underline{F}}\big\| \\
&\leqslant \frac{\epsilon}{2} + \|A_{F_0|} - A_{F|}\| < \epsilon.
\end{aligned}$$

[(5) ⇒ (2)] This is clear, since for each $F \in \mathcal{F}(S)$, $A_{F\lrcorner}$ is a bounded finite rank operator.

[(4) ⇒ (6)] Clear form the fact that a convergent net is a Cauchy net.

[(6) ⇒ (1)] If $\{A_{F|}\}_{F\in\mathcal{F}(S)}$ is a Cauchy net in $\mathcal{B}(X)$, then since each $A_{F|} \in \mathcal{K}(X)$ and since $\mathcal{K}(X)$ is complete in the operator norm, there is a $K \in \mathcal{K}(X)$ such that

$$\lim_{F\in\mathcal{F}(S)} \|A_{F|} - K\| = 0.$$

But, for each $x \in X$,

$$\lim_{F\in\mathcal{F}(S)} (A_{F|})x = \lim_{F\in\mathcal{F}(S)} A(x_{\underline{F}}) = A\Big(\lim_{F\in\mathcal{F}(S)} x_{\underline{F}}\Big) = Ax \quad \text{and} \quad \lim_{F\in\mathcal{F}(S)} (A_{F|})x = Kx.$$

Thus $Ax = Kx$ for all $x \in X$, and hence $A = K \in \mathcal{K}(X)$.

By invoking the adjoints, the equivalence of (7) to others is clear. □

Explicit examples of compact operators are given below. Note that if X has g-basis $\{e_s\}_{s\in S}$ with N-property, by part (1) of Lemma 6, $X \subseteq c_0(S)$ as sets of functions on S.

Lemma 10. *Let X have a g-basis $\{e_s\}_{s\in S}$ with N-property and suppose the collection of dual functionals $\{e_s^{\#}\}_{s\in S}$ forms a g-basis for $X^{\#}$.*

1. *For each $\xi \in c_0(S)$, define $(D_\xi x)(s) = \xi(s)x(s)$ for all $s \in S$. Then $D_\xi \in \mathcal{K}(X)$.*
2. *Let $A \in \mathcal{B}(X)$ and $\{F_n\} \subseteq \mathcal{F}(S)$ a pairwise disjoint sequence of finite subsets of S. Then $\sum_{n=1}^{\infty} \xi_n A_{\underline{F_n}} \in \mathcal{K}(X)$ (i.e., is compact) for each c_0 sequence $\{\xi_n\}$.*

3. *Assume X is $c_0(S)$ [or $\ell^p(S)$ with $1 < p < \infty$]. Let $\{\xi_n\}$ be a c_0 [resp. ℓ^p] sequence; and let $\{A_n\}$ be a bounded sequence in $\mathcal{B}(X)$ such that, for all $n \in \mathbb{N}$, $A_n = (A_n)_{\underline{F_n}}$ for some pairwise disjoint sequence $\{F_n\} \subseteq \mathcal{F}(S)$ (of finite subsets of S, i.e., $\{A_n\}$ has disjoint finite nonzero horizontal bands in the usual matrix configuration). Then $B = \sum_{n=1}^{\infty} \xi_n A_n$ converges in norm and is compact.*

By duality we also have the corresponding disjoint "vertical band" version of part (3).

Proof. (1) Let $\epsilon > 0$. Since $\xi \in c_0(s)$, there is an $F_\epsilon \in \mathcal{F}(S)$ such that

$$\|\xi - \xi_{F_\epsilon}\| = \sup_{s \in S \setminus F_\epsilon} \left|\xi(s)\right| < \epsilon.$$

Let $x \in X$ and $F \supseteq F_\epsilon$ be a finite set. We have, from Lemma 6 parts (6) and (4),

$$\left\|\left(D_\xi - (D_\xi)_{\underline{F}}\right)x\right\| = \left\|\sum_{s \in S \setminus F} \xi(s)x(s)e_s\right\| \leqslant \left\|\sum_{s \in S \setminus F} \epsilon x(s)e_s\right\| \leqslant \epsilon\|x\|.$$

Since x and F are arbitrary,

$$\left\|D_\xi - (D_\xi)_F\right\| \leqslant \epsilon \text{ for all } F_\epsilon \subseteq F \in \mathcal{F}(S).$$

Thus $D_\xi \in \mathcal{K}(X)$ by Lemma 9(3).

(2) Set $\hat{\xi}(s) = \xi_n$ for each $s \in F_n$ and $\hat{\xi}(s) = 0$ for $s \notin \bigcup_{n=1}^{\infty} F_n$. Then $\hat{\xi} \in c_0(S)$ and the sum

$$\sum_{n=1}^{\infty} \xi_n A_{\underline{F_n}} = D_{\hat{\xi}} A \in \mathcal{K}(X),$$

since $D_{\hat{\xi}} \in \mathcal{K}(X)$, which is an ideal,

(3) By assumption, there is an M such that $\|A_s\| \leqslant M$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$. There is an N such that

$$|\xi_n| < \frac{\epsilon}{M+1}, \quad \text{for all } n \geqslant N, \text{ for the } c_0 \text{ case,} \quad \text{or}$$

$$\sum_{n=N}^{\infty} |\xi_n|^p < \left(\frac{\epsilon}{M+1}\right)^p \quad \text{for all } n \geqslant N, \text{ for the case of } \ell^p.$$

Let $n > m \geqslant N$ and $x \in X$. Since $A_k x$ have pairwise disjoint supports,

$$\left\|\left[\sum_{k=m+1}^{n} \xi_k A_k\right]x\right\| = \left\|\sum_{k=m+1}^{n} \xi_k A_k x\right\| = \max_{m<k\leqslant n} \|\xi_k A_k x\| < \epsilon\|x\|$$

or for the case of ℓ^p,

$$\left\|\left[\sum_{k=m+1}^{n} \xi_k A_k\right]x\right\| = \left\|\sum_{k=m+1}^{n} \xi_k A_k x\right\| = \left[\sum_{k=m+1}^{n} \|\xi_k A_k x\|^p\right]^{1/p} < \epsilon\|x\|.$$

Thus the sequence of partial sums of the series for B is a Cauchy sequence in the (operator) norm of $\mathcal{B}(X)$. Since each partial sum is a bounded finite rank operator and hence compact, B is compact, i.e., $B \in \mathcal{K}(X)$. $\square$

4. $\mathcal{A}$-Matrices from Banach space operators

In this section we fix a Banach space X having a g-basis $\{e_s\}_{s\in S}$ with N-property, and also suppose that the set $\{e_s^\#\}_{s\in S}$ of dual functionals forms a g-basis for $X^\#$ with N-property (as defined in Section 2, Definition 5). (The spaces $\ell^p(S)$, $1 < p < \infty$, and $c_0(S)$ are examples of such spaces.) A C^*-algebra $\mathcal{A}$ with identity 1 and state space $s(\mathcal{A})$ will also be fixed. Thus $s(\mathcal{A})$ with the relative weak* topology is a compact Hausdorff space. We will consider the class $\mathcal{M}$ of all functions $A : S \times S \to \mathcal{A}$ such that the complex function $\tilde{\varphi}(A)$ defined by $(\tilde{\varphi}(A))(s,t) = \varphi(A(s,t))$ for all $s, t \in S$ induces an operator in $B(X)$, for all $\varphi \in s(\mathcal{A})$, and the map, $\varphi \mapsto \tilde{\varphi}(A)$, induced by A, from $s(\mathcal{A})$ to $\mathcal{B}(X)$ is weak* to norm continuous. We will fix X and $\mathcal{A}$ throughout this paper.

Proposition 11. *For $A \in \mathcal{M}$,*

$$\|A\| := \sup_{\varphi \in s(\mathcal{A})} \left\|\tilde{\varphi}(A)\right\| < \infty,$$

$\|\cdot\|$ is a norm on $\mathcal{M}$, and $\mathcal{M}$ is a Banach space with this norm.

Proof. Consider each element of $\mathcal{B}(X)$ as a complex-valued function on $S \times S$. Then $\mathcal{B}(X)$ with the operator norm is a Banach space of complex-valued functions on $S \times S$. Finiteness of the supremum follows from [13, Proposition 4.1]. (The finiteness of the supremum can also been established as follows. Since $s(\mathcal{A})$ with the weak* topology is a compact Hausdorff space, $C(s(\mathcal{A}), \mathcal{B}(X))$ is a Banach space; and each $A \in \mathcal{M}$ can be regarded as an element of $C(s(\mathcal{A}), \mathcal{B}(X))$.) That $\mathcal{M}$ is a Banach space follows from [13, Theorem 4.2] (it also follows from the fact that $\mathcal{M}$ is the intersection of $C(s(\mathcal{A}), \mathcal{B}(X))$ and the space of all functions A satisfying $\|A\| < \infty$). (The embedding of $\mathcal{M}$ in $C(s(\mathcal{A}), \mathcal{B}(X))$ is proper in most cases.) $\square$

Note that for $A \in \mathcal{M}$ and $G \subseteq S$, since $\tilde{\varphi}(A_{\underline{G}}) = [\tilde{\varphi}(A)]_{\underline{G}}$ for all $\varphi \in s(\mathcal{A})$, $A_{\underline{G}} \in \mathcal{M}$. Similarly, $A_{G|} \in \mathcal{M}$ and $A_{G_\lrcorner} \in \mathcal{M}$.

Proposition 12. *Let $\mathcal{K}$ denote the space of functions $A \in \mathcal{M}$ such that $\lim_{F\in\mathcal{F}} \|A - A_{\underline{F}}\| = 0$. Then $\mathcal{K}$ is a proper closed subspace of $\mathcal{M}$.*

Proof. Let $\{A_n\}$ be a sequence in $\mathcal{K}$ such that $\|A - A_n\| \to 0$ for some $A \in \mathcal{M}$. Let $\epsilon > 0$. There is an $N \in \mathbb{N}$ such that $\|A_n - A\| < \frac{\epsilon}{3}$ for all $n \geqslant N$. Since $A_N \in \mathcal{K}$, there is an $F_N \in \mathcal{F}$ such that $\|(A_N)_{\underline{F}} - A_N\| < \frac{\epsilon}{3}$ for all $F_N \subseteq F \in \mathcal{F}$. Let $F_N \subseteq F \in \mathcal{F}$.

$$\|A - A_{\underline{F}}\| \leqslant \|A - A_N\| + \big\|A_N - [A_N]_{\underline{F}}\big\| + \big\|[A_N]_{\underline{F}} - A_{\underline{F}}\big\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \big\|[A_N - A]_{\underline{F}}\big\| \leqslant \frac{2\epsilon}{3} + \|A_N - A\| < \epsilon.$$

Thus $A \in \mathcal{K}$, and hence $\mathcal{K}$ is closed.

For the proper inclusion $\mathcal{K} \subsetneq \mathcal{M}$, we note that the function $I(s,s) = 1$ (the identity of $\mathcal{A}$), and $I(s,t) = 0$ for all $s \neq t$ in S is in $\mathcal{M}$. In fact the map $\varphi \mapsto \tilde{\varphi}(I) = I$, the identity operator in $\mathcal{B}(X)$, a constant map. But $\|I - I_{\underline{F}}\| = 1$ for all $F \in \mathcal{F}$. $\square$

Proposition 13. *Let $A \in \mathcal{M}$. The following conditions are equivalent.*

1. $A \in \mathcal{K}$;
2. $\tilde{\varphi}(A) \in \mathcal{K}(X)$ *for all* $\varphi \in s(\mathcal{A})$;
3. $\lim_{F\in\mathcal{F}} \|A - A_{F|}\| = 0$;
4. $\lim_{F\in\mathcal{F}} \|A - A_{F_\lrcorner}\| = 0$.

Proof. [(1) $\Rightarrow$ (2)] Suppose $A \in \mathcal{K}$. For each $\varphi \in s(\mathcal{A})$, since

$$\lim_{F\in\mathcal{F}} \big\|\tilde{\varphi}(A) - \big[\tilde{\varphi}(A)\big]_{\underline{F}}\big\| = \lim_{F\in\mathcal{F}} \big\|\tilde{\varphi}(A - A_{\underline{F}})\big\| \leqslant \lim_{F\in\mathcal{F}} \|A - A_{\underline{F}}\| = 0,$$

$\tilde{\varphi}(A) \in \mathcal{K}(X)$. So the map $\varphi \mapsto \tilde{\varphi}(A)$ is from $s(\mathcal{A})$ to $\mathcal{K}(X)$. The norm on $\mathcal{K}(X)$ is inherited from $\mathcal{B}(X)$, the continuity follows from the membership $A \in \mathcal{K}$.

[(2) $\Rightarrow$ (4)] Let $\epsilon > 0$. For each $\varphi \in s(\mathcal{A})$, there is a weak* open set $\mathcal{V}_\varphi$ such that

$$\big\|\tilde{\psi}(A) - \tilde{\varphi}(A)\big\| < \frac{\epsilon}{3} \quad \text{for all } \psi \in \mathcal{V}_\varphi.$$

Since $s(\mathcal{A})$ is weak* compact and $s(\mathcal{A}) \subseteq \bigcup_{\varphi\in s(\mathcal{A})} \mathcal{V}_\varphi$, there are $\varphi_1, \ldots \varphi_k \in s(\mathcal{A})$ such that $s(\mathcal{A}) \subseteq \bigcup_{j=1}^k \mathcal{V}_{\varphi_j}$. Since each $\tilde{\varphi}_j(A) \in \mathcal{K}(X)$, by Lemma 9, there is an $F_j \in \mathcal{F}$ such that $\|\tilde{\varphi}_j(A) - [\tilde{\varphi}_j(A)]_{F_\lrcorner}\| < \frac{\epsilon}{3}$ for all $F_j \subseteq F \in \mathcal{F}$. Let $F_\epsilon = \bigcup_{j=1}^k F_j$. Let $F_\epsilon \subseteq F \in \mathcal{F}$, and $\psi \in s(\mathcal{A})$. There exists j such that $\psi \in \mathcal{V}_{\varphi_j}$. We have

$$\begin{aligned}
\big\|\tilde{\psi}(A - A_{F_\lrcorner})\big\| &= \big\|\tilde{\psi}(A) - \big[\tilde{\psi}(A)\big]_{F_\lrcorner}\big\| \\
&\leqslant \big\|\tilde{\psi}(A) - \tilde{\varphi}_j(A)\big\| + \big\|\tilde{\varphi}_j(A) - \big[\tilde{\varphi}_j(A)\big]_{F_\lrcorner}\big\| + \big\|\big[\tilde{\varphi}_j(A)\big]_{F_\lrcorner} - \big[\tilde{\psi}(A)\big]_{F_\lrcorner}\big\| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \big\|\tilde{\varphi}_j(A) - \tilde{\psi}(A)\big\| < \epsilon.
\end{aligned}$$

Upon taking supremum over all $\psi \in s(\mathcal{A})$, we have $\|A - A_{F_\lrcorner}\| \leqslant \epsilon$ for all $F_\epsilon \subseteq F \in \mathcal{F}$. Since ϵ is arbitrary, $\lim_{F\in\mathcal{F}} \|A - A_{F_\lrcorner}\| = 0$.

[(4) $\Rightarrow$ (2)] For each $\varphi \in s(\mathcal{A})$, since $\tilde{\varphi}(A) \in \mathcal{B}(X)$, $\tilde{\varphi}(A_{F_\lrcorner}) = [\tilde{\varphi}(A)]_{F_\lrcorner} \in \mathcal{K}(X)$ for all $F \in \mathcal{F}$ by Lemma 9. We also have

$$\lim_{F\in\mathcal{F}} \big\|\big[\tilde{\varphi}(A)\big]_{F_\lrcorner} - \tilde{\varphi}(A)\big\| = \lim_{F\in\mathcal{F}} \big\|\tilde{\varphi}(A_{F_\lrcorner} - A)\big\| \leqslant \lim_{F\in\mathcal{F}} \|A_{F_\lrcorner} - A\| = 0$$

and hence $\tilde{\varphi}(A) \in \mathcal{K}(X)$ for all $\varphi \in s(\mathcal{A})$. That is A induces a map from $s(\mathcal{A})$ to $\mathcal{K}(X)$. Since the norm on $\mathcal{K}(X)$ is inherited from $\mathcal{B}(X)$, and the map induced by A from $s(\mathcal{A})$ to $\mathcal{B}(X)$ is continuous, it is continuous from $s(\mathcal{A})$ to $\mathcal{K}(X)$.

[(3) $\Leftrightarrow$ (1)] This follows from the facts that $\|B_{\underline{F}}\| = \|[B^\#]_{F|}\|$ for all $B \in \mathcal{B}(X)$ and all $F \in \mathcal{F}$, and that an operator B is compact iff $B^\#$ is compact. $\square$

The following lemma is an immediate consequence of Lemma 10, which is about formation of $\mathcal{A}$-matrices in $\mathcal{K}$ from disjoint "horizontal bands". We omit its proof.

Lemma 14.

1. *Let $A \in \mathcal{M}$, and $\{F_n\}$ in $\mathcal{F}(S)$ be a sequence of pairwise disjoint finite subsets of S. Then for each c_0 sequence $\{\xi_n\}$, $B = \sum_{n=1}^{\infty} \xi_n A_{\underline{F_n}} \in \mathcal{K}$.*
2. *Suppose $\{A_n\colon n \in \mathbb{N}\} \subseteq \mathcal{M}$ is bounded and $A_n = (A_n)_{\underline{F_n}}$ for some pairwise disjoint $\{F_n\colon n \in \mathbb{N}\} \subseteq \mathcal{F}(S)$ (finite subsets of S), and suppose X is $c_0(S)$ or $\ell^p(S)$ with $1 < p < \infty$ and $\{\xi_n\colon n \in \mathbb{N}\}$ is in the corresponding sequence space. Then $B = \sum_{n=1}^{\infty} \xi_n A_n \in \mathcal{K}$.*

As for Lemma 10, the duality also gives us the "vertical band" version of this lemma.

5. The dual of $\mathcal{K}$

The dual space of $\mathcal{K}$ will be identified with a space of matrices with entries in $\mathcal{A}^{\#}$ in this section. It will also be shown that each element of $\mathcal{K}^{\#}$ can be extended uniquely to all of $\mathcal{M}$.

Proposition 15.

1. *Each $f \in \mathcal{K}^{\#}$ uniquely determines a function $\tilde{f} : S \times S \to \mathcal{A}^{\#}$ such that*
$$f(A) = \sum_{s\in S}\sum_{t\in S} \tilde{f}(s,t)\big(A(s,t)\big) \quad \textit{for all } A \in \mathcal{K}.$$
2. *Let $\widetilde{\mathcal{K}^{\#}} = \{\tilde{f}\colon f \in \mathcal{K}^{\#}\}$, and $\|\tilde{f}\| = \|f\|$ for each $\tilde{f} \in \widetilde{\mathcal{K}^{\#}}$. Then $\widetilde{K^{\#}}$ is isometrically isomorphic to $\mathcal{K}^{\#}$.*
3. *Suppose that for each $(s,t) \in S \times S$, there is an associated $g_{st} \in \mathcal{A}^{\#}$ such that*
$$\sum_{s\in S}\sum_{t\in S} g_{st}\big(A(s,t)\big) \quad \textit{converges for all } A \in \mathcal{K}.$$
Then
$$g(A) = \sum_{s\in S}\sum_{t\in S} g_{st}\big(A(s,t)\big) \quad \textit{for } A \in \mathcal{K}$$
defines an element $g \in K^{\#}$.

For each $a \in \mathcal{A}$, the *state norm* of a is $\|a\|_\sigma = \sup_{\varphi\in s(\mathcal{A})} |\varphi(a)|$, and it satisfies $\|a\|_\sigma \leqslant \|a\| \leqslant 2\|a\|_\sigma$ for all $a \in \mathcal{A}$ (see [13, Proposition 2.3]). Thus for $A \in \mathcal{M}$ we have
$$\big\|A(s,t)\big\| \leqslant 2\big\|A(s,t)\big\|_\sigma \leqslant 2\|A\|.$$

Proof. (1) For each $a \in \mathcal{A}$ and each $(s,t) \in S \times S$, define $E_{st}(a)$ by
$$\big(E_{st}(a)\big)(u,v) = \delta_{su}\delta_{tv}a \quad \text{for all } (u,v) \in S \times S.$$
Then, for each $\varphi \in s(\mathcal{A})$ and $(u,v) \in S \times S$,
$$\big[\tilde{\varphi}\big(E_{st}(a)\big)\big](u,v) = \varphi\big(E_{st}(u,v)\big) = \delta_{su}\delta_{tv}\varphi(a) = \big[E_{st}\big(\varphi(a)\big)\big](u,v),$$
which defines a compact operator on X, and hence $E_{st}(a) \in \mathcal{K}$ by Proposition 13 (equivalence of (2) and (1)). For $\varphi \in s(\mathcal{A})$,
$$\big\|\tilde{\varphi}\big(E_{st}(a)\big)\big\| = \big\|E_{st}\big(\varphi(a)\big)\big\| \leqslant \big|\varphi(a)\big| \leqslant \|a\|.$$
Continuity is as easily verified:
$$\big\|\tilde{\varphi}_\gamma\big(E_{st}(a)\big) - \tilde{\varphi}\big(E_{st}(a)\big)\big\| = \big|\varphi_\gamma(a) - \varphi(a)\big| \to 0$$
as φ_γ weak* converges to φ in $s(\mathcal{A})$. Define $f_{st}(a) = f(E_{st}(a))$. Then
$$\big|f_{st}(a)\big| = \big|f\big(E_{st}(a)\big)\big| \leqslant \|f\|\big\|E_{st}(a)\big\| \leqslant \|f\|\|a\|,$$
and hence $f_{st} \in \mathcal{A}^{\#}$. Let $A \in \mathcal{K}$. To simplify notation, first assume that $A(s,t) = 0$ for all $s \neq s_0$, for some $s_0 \in S$. Then since $\lim_{G\in\mathcal{F}} \|A - A_{G|}\| = 0$,
$$f(A) = \lim_{G\in\mathcal{F}} f(A_{G|}) = \lim_{G\in\mathcal{F}} f\left(\sum_{t\in G} E_{s_0 t}\big(A(s_0,t)\big)\right) = \sum_{t\in S} f_{s_0 t}\big(A(s_0,t)\big).$$
For each $A \in \mathcal{K}$, since $\lim_{F\in\mathcal{F}} \|A - A_{\underline{F}}\| = 0$ and $A_{\underline{F}} = \sum_{s\in F} A_{\underline{\{s\}}}$, we have
$$f(A) = \lim_{F\in\mathcal{F}} f(A_{\underline{F}}) = \lim_{F\in\mathcal{F}} \sum_{s\in F} f(A_{\underline{\{s\}}}) = \lim_{F\in\mathcal{F}} \sum_{s\in F}\sum_{t\in S} f_{st}\big(A(s,t)\big) = \sum_{s\in S}\sum_{t\in S} f_{st}\big(A(s,t)\big).$$

It is clear from the construction that f_{st} is uniquely determined: if

$$f(A)=\sum_{s\in S}\sum_{t\in S} g_{st}\big(A(s,t)\big)\quad \text{for all } A\in\mathcal{K},$$

then for each $s,t\in S$, $f_{st}(a)=f(E_{st}(a))=g_{st}(a)$ for all $a\in\mathcal{A}$.

(2) For each $f\in\mathcal{K}^{\#}$, define $\tilde f: S\times S\to\mathcal{A}^{\#}$ by $\tilde f(s,t)=f_{st}$ for all $s,t\in S$ as in part (1). Then the map $f\to\tilde f$ from $\mathcal{K}^{\#}$ to $\widetilde{\mathcal{K}^{\#}}$ is an isometric isomorphism with the norm as defined.

(3) For a fixed $s\in S$ and each $G\in\mathcal{F}$, let

$$g_s^G(A)=\sum_{t\in G} g_{st}\big(A(s,t)\big)\quad \text{for } A\in\mathcal{K}.$$

Then

$$\big|g_s^G(A)\big|\leqslant\sum_{t\in G}\|g_{st}\|\big\|A(s,t)\big\|\leqslant 2\|A\|\sum_{t\in G}\|g_{st}\|\quad\text{for all } A\in\mathcal{K},$$

and hence $g_s^G\in\mathcal{K}^{\#}$. Since $g_s(A):=\sum_{t\in S} g_{st}(A(s,t))$ converges by assumption, and

$$\lim_{G\in\mathcal{F}}\big|g_s^G(A)-g_s(A)\big|=0\quad\text{for each } A\in\mathcal{K},$$

there is an $M_s(A)$ such that $|g_s^G(A)|\leqslant M_s(A)$ for all $G\in\mathcal{F}$ (recall that the partial sums of a convergent sum are bounded). By the uniform boundedness principle, there is an M_s such that $\|g_s^G\|\leqslant M_s$ for all $G\in\mathcal{F}$. Thus for each $A\in K$,

$$\big|g_s(A)\big|=\lim_{G\in\mathcal{F}}\big|g_s^G(A)\big|\leqslant\limsup_{G\in\mathcal{F}}\big\|g_s^G\big\|\|A\|\leqslant M_s\|A\|,$$

and hence $g_s\in\mathcal{K}^{\#}$. A similar uniform boundedness argument shows that $g\in\mathcal{K}^{\#}$. $\square$

It follows from this that

$$\mathcal{K}^{\#}=\left\{\tilde f\colon\ \tilde f: S\times S\to\mathcal{A}^{\#},\ \text{and}\ \sum_{s\in S}\sum_{t\in S}\big(\tilde f(s,t)\big)\big(A(s,t)\big)\ \text{converges for all } A\in\mathcal{K}\right\}.$$

A uniform boundedness argument similar to that used in the proof of the last part of the preceding shows that $g\in\mathcal{M}^{\#}$ provided that the double sum converges for every $A\in\mathcal{M}$.

Recall that we are still working under the standing assumption X is a Banach space having a g-basis $\{e_s\}_{s\in S}$ with N-property and its dual space $X^{\#}$ has the set of dual functionals $\{e_s^{\#}\}_{s\in S}$ as a g-basis, also with N-property.

Proposition 16. *Suppose X is as above. For each $f\in\mathcal{K}^{\#}$, there is a unique function $\tilde f: S\times S\to\mathcal{A}^{\#}$ having the following properties.*

1. $\sum_{s\in S}\sum_{t\in S}(\tilde f(s,t))(A(s,t))=f(A)$ *for all* $A\in\mathcal{K}$.
2. *Both*

$$\hat f(A)=\sum_{s\in S}\sum_{t\in S}\big(\tilde f(s,t)\big)\big(A(s,t)\big)\quad\textit{and}$$

$$g(A)=\sum_{t\in S}\sum_{s\in S}\big(\tilde f(s,t)\big)\big(A(s,t)\big)\quad\textit{converge for all } A\in\mathcal{M}.$$

Moreover, $\hat f, g\in\mathcal{M}^{\#}$.

3. $\hat f(A)=g(A)$ *for all* $A\in\mathcal{K}$.
4. $\|\hat f\|_{\mathcal{M}^{\#}}=\|f\|_{\mathcal{K}^{\#}}=\|g\|_{\mathcal{M}^{\#}}$.
5. $\hat f(A)=g(A)$ *for all* $A\in\mathcal{M}$ *if* $X=c_0(S)$ *or* $X=\ell^p(S)$ *for* $1<p<\infty$.

Note that part (5) is a generalization of trace(AB) = trace(BA) for a trace class operator A and a bounded operator B on a Hilbert space.

Proof. Part (1) is clear from the preceding Proposition 15.

(2) Suppose the sum $\hat f(A)$ does not converge for some $A\in\mathcal{M}$. Then, since each $A_{\underline{\{s\}}}\in\mathcal{K}$, $\sum_{t\in S}(\tilde f(s,t))(A(s,t))=f(A_{\underline{\{s\}}})$ converges. Thus only the outer sum diverges. By Proposition 1, there exist an $\epsilon>0$ and a sequence $\{F_n\}$ of disjoint (finite) sets in $\mathcal{F}$ such that

$$\left|\sum_{s\in F_n}\sum_{t\in S}\big(\tilde f(s,t)\big)\big(A(s,t)\big)\right|\geqslant\epsilon.$$

Let

$$\alpha_n = \frac{1}{n}\left(\operatorname{sgn}\left(\sum_{s\in F_n}\sum_{t\in S}\big(\tilde f(s,t)\big)\big(A(s,t)\big)\right)\right)$$

(where $\operatorname{sgn}(\zeta)=\bar\zeta|\zeta|^{-1}$ if $\zeta\neq 0$ and $\operatorname{sgn}(0)=0$). Since $\{\alpha_n\}\in c_0$, by Lemma 14, $A'=\sum_{n=1}^{\infty}\alpha_n A_{\underline{F_n}}\in\mathcal{K}$. On the other hand

$$f\big(A'\big)=\sum_{n=1}^{\infty}\alpha_n\sum_{s\in F_n}\sum_{t\in S}\big(\tilde f(s,t)\big)\big(A(s,t)\big)=\sum_{n=1}^{\infty}\frac{1}{n}\left|\sum_{s\in F_n}\sum_{t\in S}\big(\tilde f(s,t)\big)\big(A(s,t)\big)\right|\geqslant\sum_{n=1}^{\infty}\frac{1}{n}\epsilon=\infty,$$

a contradiction.

Using columns instead of rows, the preceding argument can be adapted to prove that the sum $g(A)$ converges for every $A\in\mathcal{M}$.

It follows from the remark preceding this proposition that $\hat f, g\in\mathcal{M}^{\#}$.

(3) For each $A\in\mathcal{K}$, $\lim_{G\in\mathcal{F}}\|A-A_{G|}\|=0$,

$$\begin{aligned}f(A)&=\lim_{G\in\mathcal{F}}f(A_{G|})=\lim_{G\in\mathcal{F}}\sum_{s\in S}\sum_{t\in G}\big(\tilde f(s,t)\big)\big(A(s,t)\big)=\lim_{G\in\mathcal{F}}\sum_{t\in G}\sum_{s\in S}\big(\tilde f(s,t)\big)\big(A(s,t)\big)\\&=\sum_{t\in S}\sum_{s\in S}\big(\tilde f(s,t)\big)\big(A(s,t)\big)=g(A).\end{aligned}$$

(4) For each $A\in\mathcal{M}$, since $A_{\underline{F}}, A_{F|}\in\mathcal{K}$ for all $F\in\mathcal{F}$,

$$\big|g(A_{G|})\big|=\big|f(A_{G|})\big|\leqslant\|f\|_{\mathcal{K}^{\#}}\|A_{G|}\|\leqslant\|f\|_{\mathcal{K}^{\#}}\|A\|,$$

hence $|g(A)|=\lim_{G\in\mathcal{F}}|g(A_{G|})|=\lim_{G\in\mathcal{F}}|f(A_{G|})|\leqslant\|f\|_{\mathcal{K}^{\#}}\|A\|$. That is $\|g\|_{\mathcal{M}^{\#}}\leqslant\|f\|_{\mathcal{K}^{\#}}$. Since $g|_{\mathcal{K}}=f$, $\|g\|_{\mathcal{M}^{\#}}\geqslant\|f\|_{\mathcal{K}^{\#}}$. Similarly, since $\hat f(A)=\lim_{G\in\mathcal{F}}f(A_{\underline{G}})$, $\|\hat f\|_{\mathcal{M}^{\#}}=\|f\|_{\mathcal{K}^{\#}}$.

(5) Define

$$g_G(A)=\sum_{t\in G}\sum_{s\in S}\big(\tilde f(s,t)\big)\big(A(s,t)\big)\quad\text{for } A\in\mathcal{M}.$$

From earlier observations $|g_G(A)|=|g(A_{G|})|\leqslant\|g\|\|A_{G|}\|\leqslant\|g\|\|A\|$, hence $g_G\in\mathcal{M}^{\#}$. We show that $\{g_G\colon G\in\mathcal{F}\}$ is a Cauchy net in $\mathcal{K}^{\#}$. If not, then there are an $\epsilon>0$ and pairwise disjoint sequences $\{G_n\},\{H_n\}\subseteq\mathcal{F}$ and a sequence $\{A_n\}\subseteq\mathcal{M}$ such that

$$H_n\cap G_m=\emptyset\quad\forall n,m\in\mathbb{N},\qquad\|A_n\|=1,\quad\text{and}\quad(g_{G_n}-g_{H_n})(A_n)\geqslant\epsilon.$$

Let $B_n=(A_n)_{(G_n)|}-(A_n)_{(H_n)|}$. Under the assumption of $X=c_0(S)$ or $X=\ell^p(S)$ with $p\in(1,\infty)$, we have $A'=\sum_{n=1}^{\infty}\frac{1}{n}B_n\in\mathcal{K}$ converges in norm, by the dual version of Lemma 14, while

$$g\big(A'\big)=\sum_{n=1}^{\infty}\frac{1}{n}g(B_n)=\sum_{n=1}^{\infty}\frac{1}{n}(g_{G_n}-g_{H_n})(A_n)\geqslant\sum_{n=1}^{\infty}\frac{\epsilon}{n}=\infty,$$

a contradiction. We conclude that $\{g_G\colon G\in\mathcal{F}\}$ is a Cauchy net in $\mathcal{K}^{\#}$, and hence by completeness of $\mathcal{K}^{\#}$, there is an $h\in\mathcal{K}^{\#}$ such that $\lim_{F\in\mathcal{F}(S)}\|g_F-h\|=0$. Since for each $A\in\mathcal{K}$,

$$\lim_{G\in\mathcal{F}}\|A-A_{G|}\|=0,\quad\text{and}\quad\lim_{G\in\mathcal{F}}g_G(A)=\lim_{G\in\mathcal{F}}g(A_{G|})=g(A),$$

we have $g=h$ on $\mathcal{K}^{\#}$. That is $\lim_{G\in\mathcal{F}}\|g-g_G\|_{\mathcal{K}^{\#}}=0$. Thus

$$\lim_{G\in\mathcal{F}}\|\hat g-\hat g_G\|_{\mathcal{M}^{\#}}=\lim_{G\in\mathcal{F}}\|\widehat{g-g_G}\|_{\mathcal{K}^{\#}}=0.$$

For each $A\in\mathcal{M}$, since both of the double sums for $\hat f$ and g converge,

$$\begin{aligned}(\hat f-g_G)(A)&=\hat f(A)-g_G(A)\\&=\sum_{s\in S}\sum_{t\in S}\big(\tilde f(s,t)\big)\big(A(s,t)\big)-\sum_{s\in S}\sum_{t\in G}\big(\tilde f(s,t)\big)\big(A(s,t)\big)\\&=\sum_{s\in S}\sum_{t\in S\setminus G}\big(\tilde f(s,t)\big)\big(A(s,t)\big)=\sum_{s\in S}\sum_{t\in S}\big(\tilde{\tilde f}(s,t)\big)\big(A(s,t)\big)\end{aligned}$$

where $\tilde{\tilde{f}}(s,t) = \tilde{f}(s,t)$ for $t \in S \setminus G$ and 0 otherwise. From part (4), we have

$$\lim_{G\in\mathcal{F}} \|\hat{f} - g_G\|_{\mathcal{M}^\#} = \lim_{G\in\mathcal{F}} \big\|\widehat{f-(g_G)}|_{\mathcal{K}}\big\|_{\mathcal{M}^\#} = \lim_{G\in\mathcal{F}} \big\|f-(g_G)|_{\mathcal{K}}\big\|_{\mathcal{K}^\#} = \lim_{G\in\mathcal{F}} \big\|g-(g_G)|_{\mathcal{K}}\big\|_{\mathcal{K}^\#} = 0.$$

Therefore $\hat{f}(A) = \lim_{G\in\mathcal{F}} g_G(A) = g(A)$ for all $A \in \mathcal{M}$, and hence

$$\sum_{s\in S}\sum_{t\in S}\big(\tilde{f}(s,t)\big)\big(A(s,t)\big) = \hat{f}(A) = g(A) = \sum_{t\in S}\sum_{s\in S}\big(\tilde{f}(s,t)\big)\big(A(s,t)\big). \qquad \square$$

6. A decomposition of $\mathcal{M}^\#$

We see from Proposition 16 that each $f \in \mathcal{K}^\#$ has a unique extension $\hat{f}$ to all of $\mathcal{M}$ with the same norm. Denote by $\widehat{\mathcal{K}^\#}$ the subspace of $\mathcal{M}^\#$ consisting of all such extensions.

Theorem 17. *Assume $X = c_0(S)$ or $X = \ell^p(S)$ for $1 < p < \infty$. For each $f \in \mathcal{M}^\#$ there are unique $g \in \widehat{\mathcal{K}^\#}$ and $h \in \mathcal{K}^\perp$ such that $f = g + h$. Moreover the decomposition also satisfies $\|f\| = \|g\| + \|h\|$. That is $\mathcal{M}^\# = \widehat{\mathcal{K}^\#} \oplus_1 \mathcal{K}^\perp$.*

Proof. Let $f \in \mathcal{M}^\#$. Then $f|_{\mathcal{K}} \in \mathcal{K}^\#$, and hence there exists a function $\tilde{f} : S \times S \to \mathcal{A}^\#$ such that

$$f(A) = \sum_{s\in S}\sum_{t\in S}\big(\tilde{f}(s,t)\big)\big(A(s,t)\big) \quad \text{for all } A \in \mathcal{K}.$$

Let $g \in \mathcal{M}^\#$ be defined by

$$g(A) = \hat{f}(A) = \sum_{s\in S}\sum_{t\in S}\big(\tilde{f}(s,t)\big)\big(A(s,t)\big)$$

which converges for all $A \in \mathcal{M}$ by Proposition 16. Define $h = f - g$. Then $h \in \mathcal{K}^\perp$. To show the norm equality, we need only show that $\|f\| \geqslant \|g\| + \|h\|$. Let $\epsilon > 0$ be given. There are $A, B \in \mathcal{M}$ such that $\|A\| = \|B\| = 1$ and

$$g(A) > \|g\| - \frac{\epsilon}{8}, \qquad h(B) > \|h\| - \frac{\epsilon}{8}.$$

From the convergence of the double sum for $g(A)$, there is an $F_0 \in \mathcal{F}$ such that

$$\left|\sum_{S\setminus F}\sum_{t\in S}\big(\tilde{f}(s,t)\big)\big(A(s,t)\big)\right| < \frac{\epsilon}{8} \quad \forall F_0 \subseteq F \in \mathcal{F}.$$

From the finiteness of F_0 there is also a $G_0 \in \mathcal{F}$ such that

$$\left|\sum_{s\in F_0}\sum_{t\in S\setminus G_0}\big(\tilde{f}(s,t)\big)\big(A(s,t)\big)\right| < \frac{\epsilon}{8}.$$

By part (3) of Proposition 16,

$$\sum_{s\in S}\sum_{t\in S}\big(\tilde{f}(s,t)\big)\big(B(s,t)\big) \quad \text{and} \quad \sum_{t\in S}\sum_{s\in S}\big(\tilde{f}(s,t)\big)\big(B(s,t)\big),$$

both converge and are equal. Thus there is an $F_1 \in \mathcal{F}$ such that

$$F_1 \supseteq F_0 \quad \text{and} \quad \left|\sum_{s\in S\setminus F_1}\sum_{t\in S}\big(\tilde{f}(s,t)\big)\big(B(s,t)\big)\right| < \frac{\epsilon}{8}.$$

Put

$$\tilde{\tilde{f}}(s,t) = \begin{cases} 0 & \text{if } s \in F_1, \\ \tilde{f}(s,t) & \text{if } s \in S \setminus F_1. \end{cases}$$

By part (3) of Proposition 15, the function $\tilde{\tilde{f}} : S \times S \to \mathcal{A}^\#$ defines an element $\hat{f} \in \mathcal{K}^\#$. Thus

$$\sum_{s\in S\setminus F_1}\sum_{t\in S}\big(\tilde{f}(s,t)\big)\big(B(s,t)\big) = \sum_{s\in S}\sum_{t\in S}\big(\tilde{\tilde{f}}(s,t)\big)\big(B(s,t)\big) = \sum_{t\in S}\sum_{s\in S}\big(\tilde{\tilde{f}}(s,t)\big)\big(B(s,t)\big) = \sum_{t\in S}\sum_{s\in S\setminus F_1}\big(\tilde{f}(s,t)\big)\big(B(s,t)\big)$$

all converge, and hence there is a $G_1 \in \mathcal{F}$ such that $G_1 \supseteq G_0$ such that

$$\left|\sum_{t\in S\setminus G_1}\sum_{s\in S\setminus F_1}\big(\tilde{f}(s,t)\big)\big(B(s,t)\big)\right| < \frac{\epsilon}{8}.$$

Let

$$A_0(s,t)=\begin{cases} A(s,t) & \text{if } (s,t)\in F_0\times G_0,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$B_0(s,t)=\begin{cases} B(s,t) & \text{if } (s,t)\in (S\setminus F_1)\times (S\setminus G_1),\\ 0 & \text{otherwise;}\end{cases}$$

and let $C=A_0+B_0$. It is routine to check that

$$\|C\|\leqslant \max\bigl\{\|A_0\|,\|B_0\|\bigr\}\leqslant 1.$$

Since $h\in\mathcal{K}^\perp$ and $A_0, B-B_0\in\mathcal{K}$, we have $h(A_0)=0$ and $h(B)=h(B_0)$. It then follows that

$$\begin{aligned}
\|f\| &\geqslant \bigl|f(C)\bigr| = \bigl|g(A_0)+g(B_0)+h(A_0)+h(B_0)\bigr| \\
&\geqslant \bigl|g(A_0)+h(B_0)\bigr|-\bigl|g(B_0)\bigr| > \operatorname{Re}\bigl[g(A_0)\bigr]+\operatorname{Re}\bigl[h(B_0)\bigr]-\frac{\epsilon}{8} \\
&= \operatorname{Re}\Biggl[g(A)-\sum_{s\in S\setminus F_0}\sum_{t\in S}\bigl(\tilde f(s,t)\bigr)\bigl(A(s,t)\bigr)-\sum_{s\in F_0}\sum_{t\in S\setminus G_0}\bigl(\tilde f(s,t)\bigr)\bigl(A(s,t)\bigr)\Biggr]+h(B)-\frac{\epsilon}{8} \\
&> \|g\|-\frac{\epsilon}{8}-\Biggl|\sum_{s\in S\setminus F_0}\sum_{t\in S}\bigl(\tilde f(s,t)\bigr)\bigl(A(s,t)\bigr)\Biggr|-\Biggl|\sum_{s\in F_0}\sum_{t\in S\setminus G_0}\bigl(\tilde f(s,t)\bigr)\bigl(A(s,t)\bigr)\Biggr|+\|h\|-\frac{\epsilon}{4} \\
&> \|g\|+\|h\|-\frac{5\epsilon}{8} > \|g\|+\|h\|-\epsilon.
\end{aligned}$$

Since the preceding argument holds for every $\epsilon>0$, we conclude that $\|f\|\geqslant\|g\|+\|h\|$. Combining this with the obvious opposite inequality, we have $\|f\|=\|g\|+\|h\|$. $\square$

Acknowledgments

We thank the referee for the numerous suggestions for improvements, and most of all for the question that leads to the generalization to arbitrary Banach space from C^*-algebra.

Appendix A

Using C^*-algebra elements to replace the complex numbers in matrix entries seems to be much more natural than Banach space elements. We offer thanks to the referee who observed that very little of the structure of C^*-algebra is used here, apart from the fact that $\|a\|\leqslant 2\|a\|_\sigma$ for all $a\in\mathcal{A}$, and asked whether $\mathcal{A}$ could be replaced by a Banach space Y and the state space $s(\mathcal{A})$ by the unit ball $[Y^\#]_1$ of $Y^\#$. This question also leads naturally to the question of whether on specializing Y to a C^*-algebra the spaces $\mathcal{M}$ and $\mathcal{K}$ would be the same as presented above. Affirmative answers are given below.

For $f\in\mathcal{A}^\#$, f^* is the functional $f^*(a)=\overline{[f(a^*)]}$ for all $a\in\mathcal{A}$. A functional f is hermitian if $f^*=f$ [7, p. 255].

Lemma 18. *Let $\{\rho_\gamma\}_{\gamma\in\Gamma}$ be a net of hermitian linear functionals on the commutative C^*-algebra $C(\Omega)$ that weak* converges to* 0*. Then the net of positive* [*negative*] *parts* [7, *Theorem* 4.3.6, *p.* 259] $\{\rho_\gamma^+\}_{\gamma\in\Gamma}$ [*resp.* $\{\rho_\gamma^-\}_{\gamma\in\Gamma}$] *weak* converges to* 0.

Proof. Suppose not, then there is a function $f\in C(\Omega)$ such that $\rho_\gamma^+(f)\not\to 0$. By consideration of the real and imaginary parts of f, we may assume f is a real valued function. Further decomposition of f as the difference of its positive and negative parts, allows us to assume that $f\geqslant 0$. By dropping down to a subnet, if necessary, we may assume that there is an $\epsilon>0$ such that $\rho_\gamma^+(f)\geqslant 2\epsilon$ for all $\gamma\in\Gamma$. Since $\rho_\gamma^-(f)=\rho_\gamma^+(f)-\rho_\gamma(f)$ and $\rho_\gamma(f)\to 0$, we may assume that $\rho_\gamma^-(f)\geqslant\epsilon$ for all $\gamma\in\Gamma$. Regarding hermitian functionals on $C(\Omega)$ as signed measures [7, p. 219], the set $\Omega_0=\{\omega\in\Omega\colon\ f(\omega)>0\}$ has positive measures with respect to both ρ_γ^+ and ρ_γ^-, contradicting the mutual singularity of ρ_γ^+ and ρ_γ^- in the Hahn and Jordan Decomposition Theorems (see [9, p. 126]). $\square$

Lemma 19. *Let $\{\rho_\gamma\}_{\gamma\in\Gamma}$ be a net of hermitian linear functionals on $\mathcal{A}$ that is weak* convergent to linear functional ρ. Then ρ is hermitian and the net of positive* [*negative*] *parts $\{\rho_\gamma^+\}_{\gamma\in\Gamma}$* [*resp.* $\{\rho_\gamma^-\}_{\gamma\in\Gamma}$] *weak* converges to ρ^+* [*resp.* ρ^-].

Proof. That ρ is hermitian follows from definition:

$$\rho^*(a)=\overline{\bigl[\rho(a^*)\bigr]}=\overline{\Bigl[\lim_\gamma\rho_\gamma(a^*)\Bigr]}=\overline{\Bigl[\lim_\gamma\rho_\gamma^*(a^*)\Bigr]}=\overline{\Bigl[\lim_\gamma\overline{\bigl[\rho_\gamma(a)\bigr]}\Bigr]}=\lim_\gamma\rho_\gamma(a)=\rho(a)\quad\text{for all } a\in\mathcal{A}.$$

Since $\rho_\gamma - \rho \to 0$ in the weak* topology, we may assume that $\{\rho_\gamma\}_{\gamma\in\Gamma}$ weak* converges to 0. Using the canonical function representation $\Phi : \mathcal{A} \to C(s(\mathcal{A}))$ of $\mathcal{A}$ (as in [7, pp. 263–266]) given by $\hat{a}(\varphi) := (\Phi(a))(\varphi) := \varphi(a)$, $\varphi \in s(\mathcal{A})$, for each $a \in \mathcal{A}$, each hermitian functional τ on $\mathcal{A}$ corresponds to a hermitian functional $\hat{\tau}$ on $\widehat{\mathcal{A}} := \{\hat{a}\colon a \in \mathcal{A}\}$, given by $\hat{\tau}(\hat{a}) = \tau(a)$ for all $a \in \mathcal{A}$. The net $\{\widehat{\rho_\gamma}\}$ of hermitian functionals on $\widehat{\mathcal{A}}$ weak* converges to 0. Since $\widehat{\mathcal{A}} \subseteq C(s(\mathcal{A}))$ is a commutative C^*-algebra, it is isometrically *-isomorphic to $C(\Omega)$ for some compact Hausdorff space Ω. By Lemma 18, $\{\widehat{\rho_\gamma^+}\}_{\gamma\in\Gamma}$ converges to 0 in the weak* topology. Thus, with $\Phi^\# : (\widehat{\mathcal{A}})^\# \to \mathcal{A}^\#$ the Banach adjoint operator, $(\rho_\gamma^+)(a) = [\Phi^\#(\widehat{\rho_\gamma^+})](a) = \widehat{\rho_\gamma^+}(\hat{a}) \to 0$ for all $a \in \mathcal{A}$. Hence $\{\rho_\gamma^+\}$ weak* converges to 0. □

For $f \in \mathcal{A}^\#$, the real and imaginary parts of f are the hermitian functionals $\operatorname{Re} f = \frac{1}{2}(f + f^*)$ and $\operatorname{Im} f = \frac{i}{2}(f^* - f)$. Then f has the canonical representation $f = \operatorname{Re} f + i \operatorname{Im} f = (\operatorname{Re} f)^+ - (\operatorname{Re} f)^- + i(\operatorname{Im} f)^+ - i(\operatorname{Im} f)^-$, a linear combination of positive linear functionals. If we let $\varphi_1 = [(\operatorname{Re} f)^+(1)]^{-1}(\operatorname{Re} f)^+$ (with the convention $\frac{0}{0} = 0$), then $\varphi_1 \in s(\mathcal{A}) \cup \{0\}$ by [7, Theorem 4.3.2, p. 256]. Let φ_2, φ_3 and φ_4 be similarly defined for the remaining components of f. This gives rise to the canonical representation of $f = \sum_{j=1}^4 \zeta_j \varphi_j$ as a linear combination of states with $|\zeta_j| \leqslant \|f\|$, where $\zeta_1 = (\operatorname{Re} f)^+(1)$, $\zeta_2 = -(\operatorname{Re} f)^-(1)$, $\zeta_3 = i(\operatorname{Im} f)^+(1)$, $\zeta_4 = -i(\operatorname{Im} f)^-(1)$.

Proposition 20. *Let $\mathcal{A}$ be a C^*-algebra with state space $s(\mathcal{A})$ and identity 1. Let $\mathcal{Z}$ be a normed space of complex valued functions on a nonempty set S; and $\mathbf{x} : S \to \mathcal{A}$. Then $\varphi \circ \mathbf{x} \in \mathcal{Z}$ for all $\varphi \in s(\mathcal{A})$ and the map $\varphi \mapsto \varphi \circ \mathbf{x}$ is weak* to norm continuous from $s(\mathcal{A})$ to $\mathcal{Z}$ iff $f \circ \mathbf{x} \in \mathcal{Z}$ for all $f \in [\mathcal{A}^\#]_1$ and the map $f \mapsto f \circ \mathbf{x}$ is weak* to norm continuous from $[\mathcal{A}^\#]_1$ (the closed unit ball of $\mathcal{A}^\#$) to $\mathcal{Z}$.*

Proof. [⇒] Assume the map induced by $\mathbf{x}$ is weak* continuous on $s(\mathcal{A})$. Since each $f \in [\mathcal{A}^\#]_1$ is a linear combination of at most four states, $f = \sum_{j=1}^4 \alpha_j \varphi_j$, and $\mathbf{x}$ induces a continuous map from $s(\mathcal{A})$ to $\mathcal{Z}$,

$$f \circ \mathbf{x} = \left[\sum_{j=1}^4 \alpha_j \varphi_j\right] \circ \mathbf{x} = \sum_{j=1}^4 \alpha_j (\varphi_j \circ \mathbf{x}) \in \mathcal{Z}.$$

Let $\{f_\gamma\}_{\gamma\in\Gamma}$ be a net in $[\mathcal{A}]_1$ such that $f_\gamma \to f$ in the weak* topology on $[\mathcal{A}^\#]_1$. Since $f_\gamma - f \to 0$, we may thus assume without loss of generality that $f_\gamma \to 0$. Represent each

$$f_\gamma = (\operatorname{Re} f_{\gamma,1})^+ - (\operatorname{Re} f_{\gamma,2})^- + i(\operatorname{Im} f_{\gamma,3})^+ - i(\operatorname{Im} f_{\gamma,4})^- = \sum_{j=1}^4 \zeta_{\gamma,j} \varphi_{\gamma,j}$$

in the canonical representation above. By Lemma 19, the positive (negative) component of $\operatorname{Re} f$ and $\operatorname{Im} f$ weak* converges to 0. We note that under these assumptions we also have $\zeta_{\gamma,1} = (\operatorname{Re} f)^+(1) \to 0$ and likewise for $\zeta_{\gamma,j} \to 0$, $j = 2, 3, 4$. The weak* to norm continuity of the map induced by $\mathbf{x}$ on $s(\mathcal{A})$ implies that $M := \sup\{\|\psi \circ \mathbf{x}\|\colon \psi \in s(\mathcal{A})\} < \infty$. Thus,

$$\begin{aligned} 0 &\leqslant \liminf_{\gamma\in\Gamma} \|f_\gamma \circ \mathbf{x}\| \leqslant \limsup_{\gamma\in\Gamma} \|f_\gamma \circ \mathbf{x}\| = \limsup_{\gamma\in\Gamma} \left\| \left[\sum_{j=1}^4 \zeta_{\gamma,j} \varphi_{\gamma,j}\right] \circ \mathbf{x} \right\| = \limsup_{\gamma\in\Gamma} \left\| \sum_{j=1}^4 \zeta_{\gamma,j} (\varphi_{\gamma,j} \circ \mathbf{x}) \right\| \\ &\leqslant \limsup_{\gamma\in\Gamma} \left[\sum_{j=1}^4 |\zeta_{\gamma,j}| \|\varphi_{\gamma,j} \circ \mathbf{x}\|\right] \leqslant \limsup_{\gamma\in\Gamma} \left[\sum_{j=1}^4 |\zeta_{\gamma,j}| M\right] = 0, \end{aligned}$$

i.e., $\|f_\gamma \circ \mathbf{x}\| \to 0$. This shows that the map $f \to f \circ \mathbf{x}$ is weak * to norm continuous from $[\mathcal{A}^\#]_1$ to $\mathcal{Z}$.

[⇐] The converse follows immediately from the inclusion $s(\mathcal{A}) \subseteq [\mathcal{A}^\#]_1$. □

Let $\mathbf{x} : S \times S \to \mathcal{A}$. Suppose the map $\varphi \mapsto \varphi \circ \mathbf{x}$ is weak* to norm continuous from $s(\mathcal{A})$ to $\mathcal{B}(X)$, where X has g-basis $\{e_s\colon s \in S\}$. It follows from Proposition 20, that the map $f \mapsto f \circ \mathbf{x}$ is weak* to norm continuous from $[\mathcal{A}^\#]_1$ to $\mathcal{B}(X)$. All the rest of the argument carries over to the case of $\mathcal{A}$ replaced by a Banach space and $s(\mathcal{A})$ by $[Y^\#]_1$.

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